

DISTRIBUTIONAL MODES FOR SCALAR FIELD QUANTIZATION

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We propose a mode-sum formalism for the quantization of the scalar field based on distributional modes, which are naturally associated with a slight modification of the standard plane-wave modes. We show that this formalism leads to the standard Rindler temperature result, and that these modes can be canonically defined on any Cauchy surface.

I. INTRODUCTION

Manogue et al. [1] considered the quantization of the scalar field on the trousers spacetime, showing that the natural plane-wave bases are not complete in this setting. Specifically, they pointed out the existence of distributional modes which vanish identically on one region, and which correspond to initial data at the singular point joining the regions. They concluded by speculating on the possibility of quantizing this extra degree of freedom. It is then natural to consider quantizing the scalar field *entirely* in terms of such distributional modes; this work represents an attempt to do just that.

There is nothing new in using distributions to describe the scalar field. After all, the Feynman propagator is a distribution, as are the various other 2-point “functions” used, among other things, to define the notion of positive-frequency. What is different about the approach outlined here is that we choose to treat certain distributions as basis modes, which we use to expand the field for the purpose of second quantization.

Our distributional modes have *position* space indices, and in this sense are dual to the usual *momentum* space description in Minkowski space. But the duality is not precise in that the natural distributional modes correspond not to the standard plane-wave modes, but rather to what we call *pseudo* plane waves. We therefore also investigate these latter modes. We show that they are a member of an infinite family of distinct mode sets that can be used in place of the standard plane waves to derive the Rindler temperature.

To motivate our general discussion, we first consider 2-dimensional Minkowski space, explicitly constructing a family of distributional modes and discussing both their relationship to standard plane wave modes and their use in computing the Rindler temperature. This is the case that would apply to the trousers spacetime, after taking into account the periodicity the trousers impose. We then use properties of the fundamental solution of the Klein-Gordon equation to establish the existence and uniqueness of canonical distributional modes on any globally hyperbolic spacetime.

We review the standard description of the scalar field in Section II, introduce distributional modes in 2-dimensional Minkowski space in Section III, and discuss the Rindler temperature calculation in Section IV. In Section V we then show how to define canonical distributional modes on any Cauchy surface in any globally hyperbolic spacetime, and in Section VI we discuss our results.

II. BACKGROUND

In this section we review some basic properties of the quantum scalar field in Minkowski space, such as mode decomposition, positive-negative frequency decomposition, and the determination of the Fock space of states by the characterization of the vacuum.

Consider a globally hyperbolic spacetime (\mathcal{M}, g) . The classical relativistic scalar field Φ on \mathcal{M} satisfies the Klein-Gordon equation

$$(\square - m^2)\Phi = 0, \tag{1}$$

where $\square := g^{ab}\nabla_a\nabla_b$ and our signature is $(-+++)$.¹ The Klein-Gordon form for complex-valued solutions of (1) with compact Cauchy data is

$$(\Phi, \Psi)_{KG} := -i \int_{\Sigma} \left((\nabla_{\hat{n}} \bar{\Phi}) \Psi - \bar{\Phi} \nabla_{\hat{n}} \Psi \right) d\Sigma, \quad (2)$$

where $d\Sigma$ is the canonical volume element on the Cauchy surface Σ , and \hat{n} is the future pointing unit normal to Σ . This form is independent of the choice of Σ , and is more commonly recognized as the inner product for the set of “positive frequency” solutions.

The standard theory in Minkowski space uses the Fourier transform to express the field Φ as a mode-sum over plane waves,

$$\Phi(t, \vec{x}) = \int_{\vec{k}} \left(a(\vec{k}) U_{\vec{k}} + \bar{a}(\vec{k}) \bar{U}_{\vec{k}} \right) d\vec{k}, \quad (3)$$

with

$$U_{\vec{k}} = N_{\vec{k}} e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} \quad (4)$$

$$\bar{U}_{\vec{k}} = N_{\vec{k}} e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}}, \quad (5)$$

and where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. The \vec{k} -dependent normalization constants are given by $N_{\vec{k}} = \frac{1}{\sqrt{4\pi\omega_{\vec{k}}}}$. The normalization of the plane wave modes is given by

$$(U_{\vec{k}}, U_{\vec{k}})_{KG} = \delta(\vec{k} - \vec{k}) \quad (6)$$

$$(\bar{U}_{\vec{k}}, \bar{U}_{\vec{k}})_{KG} = -\delta(\vec{k} - \vec{k}) \quad (7)$$

$$(U_{\vec{k}}, \bar{U}_{\vec{k}})_{KG} = 0. \quad (8)$$

A positive-negative frequency splitting is made by choosing the $\{U_{\vec{k}}\}$ as a basis for positive frequency solutions. The positive frequency part of a (real-valued) solution is then given by

$$\Phi^+ := \int_{\vec{k}} a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t} \frac{d^3k}{\sqrt{2\omega_{\vec{k}}}(2\pi)^3} = \overline{\Phi^-}. \quad (9)$$

In quantization, the expansion coefficients $a(\vec{k})$, $\bar{a}(\vec{k})$ become operators $a(\vec{k})$, $a^\dagger(\vec{k})$ satisfying the appropriate commutation relations. The vacuum state is then characterized by the requirement

$$a(\vec{k})|0\rangle = 0, \quad \forall \vec{k}. \quad (10)$$

The final Hilbert space of states (Fock space) is generated by elements of the form [5]

$$a^\dagger(\vec{k})|0\rangle. \quad (11)$$

In a curved spacetime, there is a standard mode-sum formalism generalizing the Minkowski space procedure outlined above [5]. In fact, the only part of the above procedure that is specific to Minkowski spacetime is the choice of plane wave solutions for the mode-sum. Since there do not exist plane waves solutions to the Klein-Gordon equation in a general curved spacetime, one instead chooses a set $\{U_{\vec{k}}, \bar{U}_{\vec{k}}\}$ of complex solutions satisfying the same orthonormality conditions (6) – (8). Furthermore, the solutions must form a complete set, in that the real fields must be expandable as a mode-sum over the $\{U_{\vec{k}}, \bar{U}_{\vec{k}}\}$. Of course, finding such a set of solutions, and proving that they have the required properties to use in a mode-sum expansion, can be a very difficult problem. As we shall see, the distributional modes formalism is able to overcome these difficulties.

¹A more general form of the Klein-Gordon equation includes a term proportional to the scalar curvature. In setting this term equal to zero, we consider the standard case of minimal coupling [2]. For more on the deceptively difficult issue of nonzero coupling, see [3,4].

A. The Distributional Mode Expansion and Standard Quantum Field Theory

The distributional modes first arose in the context of the 2-dimensional flat, but topologically nontrivial, trousers spacetime [1]. We suggest that these modes (or an appropriate generalization) may similarly prove useful when considering quantization on other spacetimes that do not possess such nice properties as global hyperbolicity.² As a first step in this development, we begin with the case of two-dimensional Minkowski space \mathbb{M}^2 . In this case, the distributional modes retain the same form as in [1]. Also, the well established plane wave-based theory in Minkowski space can then be used to guide the generalization of the distributional modes theory to other spacetimes.

The modes considered by Manogue et al. satisfy the wave equation (1) in the distributional sense and are of the form

$$\Theta_y(t, x) = \frac{1}{2}[\theta(x - y + t) - \theta(x - y - t)] \quad (12)$$

$$\Delta_y(t, x) = \frac{1}{2}[\delta(x - y + t) + \delta(x - y - t)], \quad (13)$$

where $y \in \mathbb{R} \cong \Sigma := \{(0, y) \in \mathcal{M}\}$.

We consider a formalism for the canonical quantization of the scalar field based on this set of modes, which differ from the usual plane wave modes. In particular, the new modes are indexed by *position* on the surface $t = 0$, as opposed to *momentum*, are real instead of complex, and satisfy:

$$(\Theta_y, \Theta_{\tilde{y}})_{KG} = 0 \quad (14)$$

$$(\Delta_y, \Delta_{\tilde{y}})_{KG} = 0 \quad (15)$$

$$(\Theta_y, \Delta_{\tilde{y}})_{KG} = -i\delta(y - \tilde{y}). \quad (16)$$

The first step in such a formalism is to expand the field in terms of a set of basis modes, so assume³ that the field can be expanded uniquely in terms of either the plane wave modes $\{U_k, \overline{U}_k\}$ or the distributional modes $\{\Theta_y, \Delta_y\}$:

$$\Phi(x, t) = \int_k \left(a(k)U_k + \overline{a}(k)\overline{U}_k \right) dk \quad (17)$$

$$\Phi(x, t) = \int_y \left(\phi(y)\Delta_y + \pi(y)\Theta_y \right) dy. \quad (18)$$

The relationship between the expansion coefficients may then be determined by the Bogolubov coefficients

$$\begin{aligned} a(k) &= (U_k, \Phi) \\ &= \int_y dy \left(\phi(y)(U_k, \Delta_y) + \pi(y)(U_k, \Theta_y) \right) \end{aligned} \quad (19)$$

$$= \int_y dy \left[\pi(y) \left(\frac{i}{2\sqrt{\pi\omega_k}} e^{-iky} \right) + \phi(y) \left(\frac{1}{2} \sqrt{\frac{\omega_k}{\pi}} e^{-iky} \right) \right], \quad (20)$$

which can also be written in terms of a Fourier transform as

$$a(k) = \sqrt{\frac{\omega_k}{2}} \hat{\phi}(k) + i \sqrt{\frac{1}{2\omega_k}} \hat{\pi}(k). \quad (21)$$

Similarly, we find

$$\overline{a}(-k) = \sqrt{\frac{\omega_k}{2}} \hat{\phi}(k) - i \sqrt{\frac{1}{2\omega_k}} \hat{\pi}(k). \quad (22)$$

²Indeed, there has been an increasing interest in the literature with these issues [6–8], especially in connection with Hawking’s “Chronology Protection Conjecture”.

³This is a nontrivial assumption in general! See also [9].

For fixed k , $a(k)$ is simply the annihilation operator associated with a simple harmonic oscillator of frequency k with phase space variables $\hat{\phi}$ and $\hat{\pi}$! Thus, as the plane wave expansion of the field reveals the field as an infinite collection of harmonic oscillators indexed by their fixed frequency, the distributional mode expansion reveals the field as an infinite collection of Fourier transformed harmonic oscillators indexed by their fixed *position* on the $t = 0$ surface. Notice that there is Heisenberg duality at work here: the plane wave oscillators have definite frequencies, and so are completely nonlocal, whereas the distributional mode “oscillators” are localized, and have indefinite frequency. In terms of the plane wave coefficients, the distributional mode coefficients are

$$\phi(y) = i(\Theta_y, \Phi)_{KG} \quad (23)$$

$$= i \int_{\mathbb{R}} \left(a(k)(\Theta_y, U_k)_{KG} + \bar{a}(k)(\Theta_y, \bar{U}_k)_{KG} \right) dk \quad (24)$$

$$= \int_{\mathbb{R}} N_k e^{iky} a(k) + N_k e^{-iky} \bar{a}(k) dk \quad (25)$$

$$= \left[\left(\widehat{\frac{a}{\sqrt{2\omega_k}}} \right)(y) - \left(\widehat{\frac{\bar{a}}{\sqrt{2\omega_k}}} \right)(-y) \right], \quad (26)$$

where \widehat{f} represents the inverse Fourier transform of f . Similarly,

$$\pi(y) = -i(\Delta_y, \Phi)_{KG} \quad (27)$$

$$= -i \left(\left(\widehat{\sqrt{2\omega_k} a} \right)(y) - \left(\widehat{\sqrt{2\omega_k} \bar{a}} \right)(-y) \right). \quad (28)$$

The commutation relations for the operator coefficients $\phi(y)$ and $\pi(y)$ may be computed from the standard CCR imposed on the operator coefficients $a(k)$ and $a^\dagger(k)$. The result is:

$$[\phi(y), \pi(\tilde{y})] = i\delta(y - \tilde{y}), \quad (29)$$

with all other commutators vanishing which, of course, is just the equal time CCR for the field operator and the momentum operator. This is not a coincidence and is the reason for denoting the distributional mode coefficients as we have: $\phi(y) = \Phi(0, y)$, $\pi(y) = \partial_t \Phi(0, y)$. This will be addressed more fully in Section V, where the general distributional mode formalism is discussed, but we note that this validates our assumption that the field may be uniquely expanded in terms of distributional modes.

The goal is to have a mode-sum formalism that views the distributional modes and their operator coefficients as fundamental, replacing the use of plane waves. Furthermore, we want the formalism to reproduce standard flat space quantum field theory. Thus, we take the decomposition (18) as our starting point, and impose the canonical commutation relations (29) on the coefficients. The Fock space of states is generated by the vacuum, defined by (cf. (10))

$$\left(\sqrt{\frac{\omega_k}{2}} \hat{\phi}(k) + i \sqrt{\frac{1}{2\omega_k}} \hat{\pi}(k) \right) |0\rangle = 0, \quad \forall k \in \mathbb{R}. \quad (30)$$

Equation (30) defines the same vacuum as the plane waves, but in terms of a new basis.

B. Position Fock Space

One drawback of the previous construction is the definition of the vacuum via (30), which requires a Fourier transform in the variable y , which destroys the association between distributional modes and individual points on a Cauchy surface. This association is where much of the potential utility of the distributional modes lies. Further, problems can occur with the Fourier transform when working in curved spaces. However, there is an alternative description of the Fock space that does not require a Fourier transform of the (distributional mode) expansion coefficients ϕ and π , as in (30), but only a positive-negative frequency splitting of the field. The vacuum condition (10) is equivalent to the condition that the vacuum is annihilated by the positive frequency part of the field, as defined by (9):

$$\Phi^+(t, x) |0\rangle = 0 \quad \forall x \in \mathbb{R}. \quad (31)$$

Furthermore, as discussed in [10], if we denote the one-particle state of momentum k by

$$|1_k\rangle = a^\dagger(k)|0\rangle, \quad (32)$$

then the function $f_k(t, x) := \langle 0|\Phi(t, x)|1_k\rangle$ is the positive energy wave function in the configuration space version of the nonquantized Klein-Gordon equation. In this formalism, $f_k(t, x)$ is interpreted as the probability amplitude for finding a particle of momentum k at the point (t, x) in spacetime. Thus, $\Phi(t, x)|0\rangle = \Phi^-(t, x)|0\rangle$ represents a state of the field with a field quantum located at (t, x) , and is a superposition of all one-particle states $|1_k\rangle$. Similarly, the operator $\Phi^+(t, x)$ is interpreted as annihilating a particle at (t, x) . Thus, we may generate the Fock space using the “position basis” by fixing t and acting repeatedly on $|0\rangle$ with the operators $\Phi^\pm(t, x)$. To be explicit when using this position basis for Fock space, we call it “position Fock space”.

In terms of the distributional mode operators $\phi(y)$ and $\pi(y)$, the positive frequency field (at $t = 0$) that defines the vacuum via (31) is given by

$$\Phi^+(0, y) = \frac{1}{2} \left(\phi(y) + i(\Omega \star \pi)(y) \right), \quad (33)$$

or, in terms of the operator $\hat{\omega} := \sqrt{\partial_y^2 - m^2}$,

$$\Phi^+(0, y) = \frac{1}{2} \left(\phi(y) + i(\hat{\omega}^{-1} \pi)(y) \right) \quad (34)$$

where $\Omega = \widetilde{\omega_k^{-1}}(y)$, and \star denotes convolution.

We may view (34) as defining an annihilation operator $\Phi^+(0, y)$ for position Fock space by an appropriate encoding of the field data $\phi(y)$, $\pi(y)$ into the real and imaginary parts (respectively) of a complex quantity $\Phi^+(0, y)$. “Appropriate” in this case refers to equation (34), which reproduces the standard notion of positive frequency in Minkowski space. But, as is well known from the curved space viewpoint, there isn’t a unique choice of positive frequency, only preferred choices (even in Minkowski space!). Therefore, we don’t refer to (34) as *the* encoding. Indeed, we now look at a particularly convenient encoding that leads to modes called the “pseudo plane wave” modes.

C. Positive-Negative Frequency Splittings and Pseudo Plane Waves

From (34), we saw that the real part of the positive frequency field Φ^+ (at $t = 0$) involves the data ϕ , while the imaginary part involves π . We argued that we may view the forming of the ($t = 0$) positive frequency field as suitably encoding the (real) Cauchy data $\phi(y)$ and $\pi(y)$ into a single complex quantity, Φ^+ – the negative frequency part of the field being the complex conjugate. This is analogous to viewing the pair $(x, y) \in \mathbb{R}^2$ as real and imaginary parts of a single complex number z , corresponding to the positive frequency field. The negative frequency field corresponds to \bar{z} , and the positive-negative frequency decomposition $\Phi = \Phi^+ + \Phi^-$ corresponds to the statement $2x = z + \bar{z}$.

The standard Fourier transform formalism leads to a particular encoding of the data $\phi(y)$ and $\pi(y)$ by the plane wave expansion of Φ^+ , that of (34). However, the simplest procedure would be to use the above analogy with \mathbb{R}^2 more directly and define

$$2\Phi_{ppw}^+(0, x) := \phi(x) + i\pi(x). \quad (35)$$

Then, we would have $2\Phi_{ppw}^-(0, x) := \phi(x) - i\pi(x)$. To see how this might arise from a mode decomposition, consider the “pseudo plane wave” modes V_k and \bar{V}_k , defined by

$$V_k := \frac{1}{2\sqrt{\pi}} e^{ikx} \left(\cos(\omega_k t) - \frac{i}{\omega_k} \sin(\omega_k t) \right) \quad (36)$$

$$\bar{V}_k = \frac{1}{2\sqrt{\pi}} e^{-ikx} \left(\cos(\omega_k t) + \frac{i}{\omega_k} \sin(\omega_k t) \right). \quad (37)$$

These modes satisfy the same normalization conditions as the plane waves:

$$(V_k, V_{\tilde{k}})_{KG} = \delta(k - \tilde{k}) \quad (38)$$

$$(\bar{V}_k, \bar{V}_{\tilde{k}})_{KG} = -\delta(k - \tilde{k}) \quad (39)$$

$$(V_k, \bar{V}_{\tilde{k}})_{KG} = 0. \quad (40)$$

Expanding the field in terms of the pseudo plane waves yields

$$\Phi(t, x) = \int_k \left(b(k) V_k + \bar{b}(k) \bar{V}(k) \right) dk. \quad (41)$$

The Bogolubov transformation between the pseudo plane waves and distributional modes, together with the transformation between the pseudo plane waves and the plane waves, imply:

$$b(k) = \frac{1}{\sqrt{2}} \left(\tilde{\phi}(k) + i\tilde{\pi}(k) \right) \quad (42)$$

$$\bar{b}(k) = \frac{1}{\sqrt{2}} \left(\tilde{\phi}(-k) - i\tilde{\pi}(-k) \right) \quad (43)$$

$$b(k) = \frac{1}{2\sqrt{\omega_k}} \left((1 + \omega_k) a(k) + (1 - \omega_k) \bar{a}(-k) \right) \quad (44)$$

$$\bar{b}(k) = \frac{1}{2\sqrt{\omega_k}} \left((1 - \omega_k) a(-k) + (1 + \omega_k) \bar{a}(k) \right). \quad (45)$$

The first two of these equations show that the pseudo plane wave coefficients are related to the Fourier transform of the Cauchy data $\phi(y)$ and $\pi(y)$ in essentially the same manner as in (35). Thus, for y -space pseudo plane waves, one could either Fourier transform on the variable k , or take the appropriate complex combination of the distributional modes to obtain the desired expansion:

$$V_y = \frac{1}{\sqrt{2}} (\Delta_y - i\Theta_y). \quad (46)$$

Another advantageous property of the pseudo plane waves was noted by Dray and Manogue in [11]. In the case of the massless scalar field, the zero frequency limit, $\lim_{k \rightarrow 0} U_k$, does not exist for the plane wave modes, since the normalization factor N_k diverges. Even if we take the zero frequency limit before normalizing and define $U_0 = \lim_{k \rightarrow 0} e^{-i\omega_k t + i k x} = 1$, it is real (and hence has no notion of positive frequency) and is orthogonal to the modes $U_{\vec{k}}, \bar{U}_{\vec{k}}$, as well as to itself. Thus, the set $\{U_0, U_{\vec{k}}, \bar{U}_{\vec{k}}\}$ is degenerate with respect to the Klein-Gordon form. It is common in practice to simply discard this mode as unphysical. However, this is hard to justify in spacetimes with compact spatial sections, since in this case, U_0 is a smooth function with compact support. In contrast, Dray and Manogue pointed out that V_k and \bar{V}_k *do* have well defined zero frequency limits, namely $\frac{1}{\sqrt{2}}(1 - i)$ and $\frac{1}{\sqrt{2}}(1 + i)$. It is also noteworthy that the distributional modes Θ_y and Δ_y also have zero norm, and that $\{U_0, \Theta_y, \Delta_y\}$ is *not* degenerate with respect to the Klein-Gordon product. This is an explicit example of how the distributional modes (as well as the pseudo plane wave modes) have the ability to capture all the degrees of freedom of a system, where the plane waves cannot. The pseudo plane waves are one example of a family of sets of modes we call “generalized plane waves”. We elaborate further on this subject in the next section, after discussing distributional modes in Rindler space.

IV. RINDLER SPACE

A. Quantum Field Theory in Rindler Space

The Rindler wedge [5,11] is the globally hyperbolic, open submanifold of Minkowski space corresponding to the right wedge $x > |t|$. Canonical “Rindler coordinates” (τ, ρ) are defined in terms of Minkowski coordinates as follows:

$$\tau = \frac{1}{a} \tanh^{-1} \left(\frac{t}{x} \right) \quad (47)$$

$$\rho = \frac{1}{2a} \ln[a^2(x^2 - t^2)] \quad (48)$$

$$t = \frac{e^{a\rho}}{a} \sinh(a\tau) \quad (49)$$

$$x = \frac{e^{a\rho}}{a} \cosh(a\tau), \quad (50)$$

where $a > 0$ is a constant. The line element is given by:

$$ds^2 = e^{2a\rho}(-d\tau^2 + d\rho^2). \quad (51)$$

The rays $\tau = C$, C a constant, are acceptable Cauchy surfaces for the Rindler wedge. For definiteness we will work with the Cauchy surface Σ_R , defined by $\tau = 0$. The curves $\rho = D$, D a constant, are hyperbolae symmetric about the x -axis and have as asymptotes the lines $x = \pm t$. These hyperbolae represent the world lines of observers undergoing a constant acceleration $ae^{-a\rho}$.

In the case of the massless scalar field ($m = 0$), the Klein-Gordon equation is conformally invariant, and the line element (51) is conformal to the Minkowski line element [1,5]. It follows that the entire quantization procedure in terms of Rindler plane waves will be formally identical to the Minkowski space quantization procedure, but with ρ replacing x and τ replacing t . Thus, we will have an orthonormal basis of positive frequency solutions (with respect to Rindler time τ), $U_l^R(\tau, \rho) = \frac{1}{\sqrt{4\nu_l\pi}} e^{-i\nu_l\tau + il\rho}$, that will split Rindler space fields (i.e. solutions to the Rindler space wave equation) into positive and negative frequency parts, $\Phi^R(\tau, \rho) = \Phi^{R+} + \Phi^{R-}$. In the usual way, the coefficients $a^R, a^{R\dagger}$ of the field expansion become operators satisfying the CCR (79) and define a vacuum state $|0^R\rangle$ which serves as a cyclic vector for a Fock space of states \mathcal{F}^R .

Comparing the Minkowski vacuum $|0\rangle$ and the Rindler vacuum $|0^R\rangle$, one finds [5]:

$$\langle 0|a^{R\dagger}a^R|0\rangle = \frac{1}{e^{\frac{2\pi\nu}{a}} - 1}. \quad (52)$$

Thus, the Minkowski vacuum state contains Rindler particles (i.e. the notion of particles defined by quantum field theory the Rindler space) in each mode ν , at a density $1/(e^{\frac{2\pi\nu}{a}} - 1)!$ Moreover, this spectrum of particles is precisely a Planckian spectrum at temperature $\frac{a}{2\pi}$. As mentioned above, curves of the form “ $\rho = D$ ” represent uniformly accelerating observers, and so we may interpret the difference in vacua by saying that uniformly accelerating observers in Minkowski space with a quantum field in the vacuum state will “feel themselves immersed in a thermal bath of radiation at the temperature $\frac{a}{2\pi}$ ” [12]. This is the Unruh effect.

B. Distributional Modes in Rindler Space

The Rindler space distributional and pseudo plane wave modes are *formally* the same as the Minkowski versions, but in terms of different variables:

$$\Theta_\lambda(\tau, \rho) = \frac{1}{2}(\theta(\rho - \lambda + \tau) - \theta(\rho - \lambda - \tau)) \quad (53)$$

$$\Delta_\lambda(\tau, \rho) = \frac{1}{2}(\delta(\rho - \lambda + \tau) + \delta(\rho - \lambda - \tau)) \quad (54)$$

$$V_l^R = \frac{1}{\sqrt{2}}(\tilde{\Theta}_l - i\tilde{\Delta}_l), \quad (55)$$

where $\lambda, l \in \Sigma^R$. The distributional modes Θ_y are coordinate independent, in the sense that $\Theta_\lambda(\tau, \rho) = \Theta_{y(\lambda)}(t(\tau, \rho), x(\tau, \rho))$. However, the same does not hold for Δ_λ and Δ_y . The modes Δ_λ (respectively, Δ_y) have been defined as the τ (respectively t) derivative of Θ_λ (respectively Θ_y), and $\partial_\tau \neq \partial_t$. This is the origin of the inequivalent notions of positive frequency with respect to Minkowski and Rindler coordinates in the distributional modes formalism, where it appears as different notions of time (t vs. τ).

In the right Rindler wedge, we have the expansions

$$\Phi^R(\rho, \tau) = \int_l dl (a^R(l)U_l^R + \overline{a^R(l)}\overline{U_l^R}) \quad (56)$$

$$\Phi^R(\rho, \tau) = \int_l dl (b^R(l)V_l^R + \overline{b^R(l)}\overline{V_l^R}) \quad (57)$$

$$\Phi^R(\rho, \tau) = \int_\lambda d\lambda (\phi^R(\lambda)\Delta_\lambda + \pi^R(\lambda)\Theta_\lambda). \quad (58)$$

As in the Minkowski case, it follows from the computation of the Bogolubov transformations that the expansion coefficients, hence the annihilation and creation operators defined by the modes, are related as follows:

$$a^R(l) = \frac{1}{2\sqrt{\nu}} \left(\nu \tilde{\phi}^R(l) + i \tilde{\pi}^R(l) \right) \quad (59)$$

$$\overline{a^R}(l) = \frac{1}{2\sqrt{\nu}} \left(\nu \tilde{\phi}^R(-l) + i \tilde{\pi}^R(-l) \right) \quad (60)$$

$$b^R(l) = \frac{1}{\sqrt{2}} \left(\tilde{\phi}^R(l) + i \tilde{\pi}^R(l) \right) \quad (61)$$

$$\overline{b^R}(l) = \frac{1}{\sqrt{2}} \left(\tilde{\phi}^R(-l) - i \tilde{\pi}^R(-l) \right) \quad (62)$$

$$b^R(l) = \frac{1}{2\sqrt{\nu_l}} \left((1 + \nu_l) a^R(l) + (1 - \nu_l) \overline{a^R}(-l) \right) \quad (63)$$

$$\overline{b^R}(l) = \frac{1}{2\sqrt{\nu_l}} \left((1 - \nu_l) a^R(-l) + (1 + \nu_l) \overline{a^R}(l) \right). \quad (64)$$

The sets of modes U_l , V_l , and $\Theta_\lambda, \Delta_\lambda$ are related by

$$V_l = \left(\frac{1}{2} + \frac{1}{2\omega_k} \right) U_k + \left(\frac{1}{2} - \frac{1}{2\omega_k} \right) \overline{U}_k \quad (65)$$

$$= \frac{1}{\sqrt{2}} (\tilde{\Theta}_l - i \tilde{\Delta}_l). \quad (66)$$

C. The Unruh Effect

A standard method for deriving the Unruh and Hawking effects is to analyze the Bogolubov transformations between the basis modes associated with the two different coordinate systems corresponding to two sets of observers **a** and **b**. That one obtains a *thermal* spectrum of “**a** particles” in the **b** vacuum state follows from a particular property (say, P) satisfied by the Bogolubov coefficients α_{lk} and β_{lk} . For the Unruh effect, one makes the standard choice of basis modes in both coordinate systems – plane waves. Although this is a natural choice to make, the point has been stressed that the consideration of alternatives to the plane waves may be advantageous and, in certain spacetimes, even necessary. Thus, the question arises if, and to what extent, the properties of the Bogolubov coefficients that lead to the thermal spectrum result do (or do not) depend on this choice of modes.

In the remainder of this section we will define a family of Bogolubov transformations on plane waves to obtain an infinite family F of sets of “generalized plane wave” modes. We will regard each set in the family as an alternative set of basis modes for the mode-sum formulation of quantum field theory. Considering both Minkowski and Rindler generalized plane waves, we show that the Unruh effect may be derived by choosing any set of modes in the family to serve as the basis modes, so long as corresponding sets of generalized plane waves are used in the two different coordinate systems.

We give a brief review of the derivation of the Unruh effect here in §IV C, pointing out the property P of the Bogolubov coefficients that leads to the Unruh effect. In §IV D and §IV E we define the family F of sets of modes and prove the invariance of the property P under changes of mode sets within F.

We compute the expression for the Bogolubov coefficients α_{lk} and β_{lk} for the Minkowski plane waves $U_{\vec{k}}$ and U_l^R .

$$\alpha_{lk} = (U_l^R, U_{\vec{k}})_{KG} \quad (67)$$

$$= -i \int_0^\infty \left(\frac{1}{\sqrt{4\pi\omega_k}} \right) \left(\frac{1}{\sqrt{4\pi\nu_l}} \right) \left(i\nu_l e^{ikx} e^{-a\rho(x) - i l \rho(x)} + i\omega_k e^{ikx} e^{-i l \rho(x)} \right) dx \quad (68)$$

$$= \frac{\sqrt{\omega_k}}{4\pi\sqrt{\nu_l}} \int_0^\infty e^{ikx} e^{-\frac{il}{a} \ln(ax)} \left(1 + \frac{\nu_l}{ax\omega_k} \right) dx. \quad (69)$$

Let $x \rightarrow iy$ and rotate the resulting contour back to the positive real axis $x > 0$. Using $\ln(iy) = \frac{i\pi}{2} + \ln(y)$, $y \in \mathbb{R}$, we get

$$\alpha_{lk} = \frac{i\sqrt{\omega_k}}{4\pi\sqrt{\nu_l}} \int_0^\infty e^{-ky} e^{-\frac{il}{a} \ln(iay)} \left(1 + \frac{\nu_l}{iay\omega_k} \right) dy \quad (70)$$

$$= e^{\frac{i\pi}{2a}} \left[\frac{i\sqrt{\omega_k}}{4\pi\sqrt{\nu_l}} \int_0^\infty e^{-ky} e^{-\frac{il}{a} \ln(ay)} \left(1 + \frac{\nu_l}{iay\omega_k} \right) dy \right]. \quad (71)$$

By essentially the same calculation, one finds

$$-\beta_{lk} = (U_l^R, \overline{U}_{\tilde{k}})_{KG} \quad (72)$$

$$= e^{-\frac{l\pi}{2a}} \left[-\frac{i\sqrt{\omega_k}}{4\pi\sqrt{\nu_l}} \int_0^\infty e^{-ky} e^{\frac{il}{a} \ln(ay)} \left(-1 + \frac{\nu_l}{iay\omega_k} \right) dy \right]. \quad (73)$$

Thus, we see that the Bogolubov coefficients satisfy the property

$$\alpha_{lk} = e^{\frac{l\pi}{a}} \overline{\beta}_{lk}, \quad (74)$$

which implies

$$|\alpha_{lk}|^2 = e^{\frac{2l\pi}{a}} |\overline{\beta}_{lk}|^2. \quad (75)$$

This is the property P from which the thermal spectrum result follows, provided the Bogolubov identities [5,9] are satisfied.

D. Generalized Plane Waves

Consider a Bogolubov transformation of the plane waves

$$\begin{pmatrix} U_k \\ \overline{U}_{-k} \end{pmatrix} \mapsto \begin{pmatrix} V_k \\ \overline{V}_{-k} \end{pmatrix} = \begin{pmatrix} A_k & B_k \\ B_k & A_k \end{pmatrix} \begin{pmatrix} U_k \\ \overline{U}_{-k} \end{pmatrix}, \quad (76)$$

that does not mix frequencies and is subject to

$$\det \begin{pmatrix} A_k & B_k \\ B_k & A_k \end{pmatrix} = A_k^2 - B_k^2 = 1, \quad (77)$$

so that the normalization is preserved. Generally, we have such a transformation for each function $\theta(k)$ if we define $A_k = \cosh(\theta(k))$, $B_k = \sinh(\theta(k))$. If $B_k \neq 0$, there will be mixing of positive and negative frequencies, and the transformed modes will define a different vacuum state than the standard Minkowski vacuum [5]. An explicit example of such a transformation is given by the pseudo plane wave modes, for which $\theta(k) = -\frac{1}{2} \ln(\omega_k)$. We may formally reproduce the above construction for plane waves in Rindler space and write the transformation as

$$\begin{pmatrix} V_l^R \\ \overline{V}_{-l}^R \end{pmatrix} = \begin{pmatrix} A_l & B_l \\ B_l & A_l \end{pmatrix} \begin{pmatrix} U_l^R \\ \overline{U}_{-l}^R \end{pmatrix} = \begin{pmatrix} A_l U_l^R + B_l \overline{U}_{-l}^R \\ B_l U_l^R + A_l \overline{U}_{-l}^R \end{pmatrix}. \quad (78)$$

We now have a new set of modes $\{V_k, \overline{V}_{-k}\}$ in Minkowski space and a new set $\{V_l^R, \overline{V}_{-l}^R\}$ in Rindler space.

By using a different set of basis modes we obtain a different set of operator coefficients. It is straightforward to check that for the above class of transformations the new operator coefficients satisfy the same canonical commutation relations as their plane wave counterparts. That is,

$$[a(k), a^\dagger(\tilde{k})] = \delta(k - \tilde{k}), \quad (79)$$

together with

$$b(k) = A_k a(k) + B_k a^\dagger(-k) \quad (80)$$

$$b^\dagger(k) = A_k a^\dagger(k) + B_k a(-k), \quad (81)$$

and $A_k^2 - B_k^2 = 1$ imply that

$$[b(k), b^\dagger(\tilde{k})] = \delta(k - \tilde{k}) \quad (82)$$

(with all other commutators vanishing). Thus, the commutation relations (79) are unchanged by the transformation (80),(81).

Using (80),(81), we may compute the following vacuum expectation values:

$$\begin{aligned}
\langle 0|b^\dagger(k)b(\tilde{k})|0\rangle &= \langle 0|\{A_k a^\dagger(k) + B_k a(-k)\}\{A_{\tilde{k}} a(\tilde{k}) + B_{\tilde{k}} a^\dagger(-\tilde{k})\}|0\rangle \\
&= \langle 0|B_k B_{\tilde{k}} a(-k)a^\dagger(-\tilde{k})|0\rangle \\
&= B_k B_{\tilde{k}} \langle 0|\delta(\tilde{k} - k) + a^\dagger(-\tilde{k})a(-k)|0\rangle \\
&= B_k B_{\tilde{k}} \langle 0|\delta(\tilde{k} - k)|0\rangle + \langle 0|a^\dagger(-\tilde{k})a(-k)|0\rangle \\
&= B_k^2 \delta(\tilde{k} - k).
\end{aligned} \tag{83}$$

Consider the special case where the operators $a(k), a^\dagger(k)$ are the usual Minkowski plane wave operator coefficients and $b(k), b^\dagger(k)$ are the pseudo plane wave operator coefficients. The number operator for k -mode particles is $b^\dagger(k)b(k)$. The vacuum expectation value of the number operator for k -mode particles is then infinite:

$$\langle 0|b^\dagger(k)b(k)|0\rangle \rightarrow \frac{1}{4}\delta(0)\left(\frac{1}{\omega_k} + \omega_k - 2\right). \tag{84}$$

The infinity due to the delta function arises because of the infinite spatial volume of time slices ($t = k, k \in \mathbb{R}$ hypersurfaces) in Minkowski space. In particular, if we “put the field in a box” (that is, assume that the time slices are flat 3-tori instead of copies of \mathbb{R}^3 [13]), the delta function becomes a Kronecker delta, and in this case,

$$\langle 0|b^\dagger(k)b(k)|0\rangle = \frac{1}{4}\left(\frac{1}{\omega_k} + \omega_k - 2\right). \tag{85}$$

The total particle content still diverges:

$$\sum_{k=-\infty}^{\infty} \langle 0|b^\dagger(k)b(k)|0\rangle = \frac{1}{4} \sum_{k=-\infty}^{\infty} \left(\frac{1}{\omega_k} + \omega_k - 2\right) = \infty. \tag{86}$$

This shows explicitly that the vacuum defined by the pseudo plane wave modes defines a Fock space that is not unitarily equivalent to the Fock space generated by the standard Minkowski vacuum, for finiteness of $\sum_{k=-\infty}^{\infty} \langle 0|b^\dagger(k)b(k)|0\rangle$ is necessary and sufficient to guarantee that the pseudo plane wave vacuum and the standard Minkowski vacuum define unitarily equivalent Fock spaces [2].

The following section shows that, somewhat surprisingly, the modes $\{V_k, \overline{V}_{-k}\}$ and $\{V_l^R, \overline{V}_{-l}^R\}$ can nevertheless be used to derive the Unruh effect.

E. Invariance Theorem

We have already noted that the if property P is satisfied by the Bogolubov coefficients, then the Unruh effect follows via the consistency conditions satisfied by the Bogolubov coefficients. We first derive some properties of the Bogolubov transformations α_{lk} and β_{lk} between Minkowski and Rindler plane waves.

Lemma 1 *The Bogolubov coefficients α_{lk} and β_{lk} associated with the plane wave modes in Minkowski and Rindler space satisfy*

$$\overline{\alpha}_{lk} = \alpha_{-l-k} \tag{87}$$

$$\overline{\beta}_{lk} = \beta_{-l-k}. \tag{88}$$

Proof:

By direct computation,

$$\overline{\alpha}_{lk} = \overline{(U_l^R U_{\tilde{k}})}_{KG} \tag{89}$$

$$= i \int_0^\infty \left(\frac{1}{\sqrt{4\pi\omega_k}}\right) \left(\frac{1}{\sqrt{4\pi\nu_l}}\right) e^{-ikx} e^{il\rho(x)} \left(-i\nu_l e^{-a\rho(x)} - i\omega_k\right) dx \tag{90}$$

$$= -\left(\frac{1}{\sqrt{4\pi\omega_k}}\right) \left(\frac{1}{\sqrt{4\pi\nu_l}}\right) \int_0^\infty e^{i(l\rho(x)-kx)} \left(\nu_l e^{-a\rho(x)} + \omega_k\right) dx \tag{91}$$

$$= \alpha_{-l-k}. \tag{92}$$

Similarly,

$$\bar{\beta}_{lk} = -(U_l^R, \bar{U}_k)_{KG} \quad (93)$$

$$= -i \left(\frac{1}{\sqrt{4\pi\omega_k}} \right) \left(\frac{1}{\sqrt{4\pi\nu_l}} \right) \int_0^\infty e^{-i(l\rho(x)+kx)} \left(-i\nu_l e^{-a\rho(x)} - i\omega_k \right) dx \quad (94)$$

$$= - \left(\frac{1}{\sqrt{4\pi\omega_k}} \right) \left(\frac{1}{\sqrt{4\pi\nu_l}} \right) \int_0^\infty e^{i(-l\rho(x)-kx)} \left(\nu_l e^{-a\rho(x)} + \omega_k \right) dx \quad (95)$$

$$= \beta_{-l-k}. \quad (96)$$

We now prove the main result.

Proposition 1 *Property P holds for the transformation $\{V_k, \bar{V}_{-k}\} \mapsto \{V_l^R, \bar{V}_{-l}^R\}$ if and only if P holds for the transformation $\{U_k, \bar{U}_{-k}\} \mapsto \{U_k^R, \bar{U}_{-k}^R\}$. That is, for $\tilde{\alpha}_{lk} = (V_l^R, V_k)_{KG}$, $\tilde{\beta}_{lk} = -(V_l^R, \bar{V}_k)_{KG}$, $\alpha_{lk} = (U_l^R, U_k)_{KG}$ and $\beta_{lk} = -(V_l^R, \bar{V}_k)_{KG}$, then $\tilde{\alpha}_{lk} = e^{\frac{\pi l}{a}} \bar{\beta}_{lk}$ if and only if $\alpha_{lk} = e^{\frac{\pi l}{a}} \bar{\beta}_{lk}$.*

Proof:

The Bogolubov coefficients $\tilde{\alpha}_{lk}$ may be expressed in terms of the (untransformed) plane wave Bogolubov coefficients by substitution:

$$\tilde{\alpha}_{lk} = (V_l^R, \bar{V}_{-k})_{KG} \quad (97)$$

$$= (A_l U_l^R + B_l \bar{U}_{-l}^R, A_k U_k + B_k \bar{U}_{-k})_{KG} \quad (98)$$

$$= A_l A_k (U_l^R, U_k)_{KG} + A_l B_k (U_l^R, \bar{U}_{-k})_{KG} + \quad (99)$$

$$B_l A_k (\bar{U}_{-l}^R, U_k)_{KG} + B_l B_k (\bar{U}_{-l}^R, \bar{U}_{-k})_{KG}. \quad (100)$$

Using the lemma, we get

$$\tilde{\alpha}_{lk} = A_l A_k \alpha_{lk} - A_l B_k \bar{\beta}_{-lk} + B_l A_k \bar{\beta}_{-lk} - B_l B_k \alpha_{lk} \quad (101)$$

$$= (A_l A_k - B_l B_k) \alpha_{lk} + (B_l A_k - A_l B_k) \bar{\beta}_{-lk}. \quad (102)$$

Assuming $\alpha_{lk} = e^{\frac{l\pi}{a}} \bar{\beta}_{lk}$, we may use this in the above equation to obtain

$$\tilde{\alpha}_{lk} = (A_l A_k - B_l B_k) e^{\frac{l\pi}{a}} \bar{\beta}_{lk} + (B_l A_k - A_l B_k) \bar{\beta}_{-lk}. \quad (103)$$

Similarly, one finds

$$\tilde{\beta}_{lk} = (A_l A_k - B_l B_k) \bar{\beta}_{lk} + (B_l A_k - A_l B_k) e^{-\frac{l\pi}{a}} \bar{\beta}_{-lk}. \quad (104)$$

By comparing (103) and (104), we conclude that

$$\tilde{\alpha}_{lk} = e^{\frac{l\pi}{a}} \bar{\beta}_{lk}, \quad (105)$$

that is, the Bogolubov coefficients for the transformed modes possess the property P. Applying the same argument to the inverse transformation proves the converse.

V. GENERAL THEORY OF DISTRIBUTIONAL MODES

A. Existence and Uniqueness of Canonical Distributional Modes

We have so far defined distributional modes explicitly for 2-dimensional Minkowski and Rindler spaces, and the generalization to n -dimensional Minkowski and Rindler spacetimes is straightforward. However, our desire is to define distributional modes in general, curved spacetimes, and the key to this issue lies in the fact that the distributional modes are closely related to the *fundamental solution* of the Klein-Gordon equation [14,15]. This observation makes it possible to establish the existence and uniqueness of a canonical type of distributional mode.

The family of distributional modes $\{\Theta_y\}$, taken as a single distribution, corresponds precisely to the restriction to Σ of the fundamental solution (the difference between the advanced and retarded Green functions) $G(x, y)$ of the Klein-Gordon equation ⁴:

$$\Theta_y = G(x, y)|_{y \in \Sigma}. \quad (106)$$

An important result concerning the fundamental solution is that it is uniquely defined on any globally hyperbolic spacetime [14,15]. Thus, the distributional modes $\{\Theta_y\}$ have a unique, coordinate invariant generalization to arbitrary globally hyperbolic spacetimes that requires, and will depend on, only a choice of Cauchy surface. The distributional modes $\{\Delta_y\}$ were defined as the (coordinate) time derivative of the modes $\{\Theta_y\}$, as were the Rindler versions. As mentioned in §IV B, the Minkowski and Rindler versions of these modes differed because of the different definitions of time coordinate. So, although there is a unique set of modes $\{\Theta_y\}$ associated to a given Cauchy surface in a globally hyperbolic spacetime, this will not be the case for $\{\Delta_y\}$.

The Minkowski space distributional modes have the property that their nonzero data, being delta functions, have support concentrated on a single point. For example, in Minkowski space, the distributional modes have the data (at $t = 0$)

$$\{\Theta_y, \Delta_y\} = \left\{ \begin{pmatrix} 0 \\ \delta(x - y) \end{pmatrix}, \begin{pmatrix} \delta(x - y) \\ 0 \end{pmatrix} \right\}. \quad (107)$$

It is this property that compelled their study, since it allows one to isolate degrees of freedom associated with points in spacetime. Thus, the distributional mode formulation of quantum field theory would be the most natural approach to problems of quantization in spacetimes where there are singular points, such as the trousers spacetime. With this in mind, we define *canonical* distributional modes that generalize the Minkowski distributional modes in the following sense. For Cauchy surfaces in Minkowski space of the form $t = C$, C a constant, where (t, \vec{x}) are inertial coordinates, the normal vector coincides with ∂_t . Since every Cauchy surface in a globally hyperbolic spacetime has a unique, future pointing normal vector, we may generalize the Minkowski definition of distributional modes using the normal vector to define a preferred set of modes Δ_y .

Definition V.1 *Let (\mathcal{M}, g) be a globally hyperbolic n -dimensional spacetime and Σ a Cauchy surface. Let x and y be 2 independent copies of a local coordinate for \mathcal{M} . Restrict to $y \in \Sigma$. The canonical distributional modes are defined to be the distributional solutions to the Klein-Gordon equation, Θ_y and Δ_y , corresponding to the data*

$$\{\Theta_y, \Delta_y\} = \left\{ \begin{pmatrix} 0 \\ \delta(x - y) \end{pmatrix}, \begin{pmatrix} \delta(x - y) \\ 0 \end{pmatrix} \right\}. \quad (108)$$

This definition is substantiated by the following theorem:

Proposition 2 *Let (\mathcal{M}, g) be a globally hyperbolic n -dimensional spacetime and Σ a Cauchy surface. Then there exist a unique set of distributional modes $\{\Theta_y, \Delta_y\}$. Furthermore, the set of modes is complete in the sense that any C^∞ solution of the Klein-Gordon equation $(\square - m^2)\Phi = 0$ with compact spatial support may be uniquely expanded in terms of the distributional modes.*

Proof:

The solution to the initial value problem for the Klein-Gordon equation is given in terms of the homogeneous fundamental solution $G(x, y)$ by

$$\iota: \begin{pmatrix} \phi \\ \pi \end{pmatrix} \mapsto \int_{\Sigma} \left(\pi(y)G(x, y) - \phi(y)\nabla_n^y G(x, y) \right) d\Sigma^y = \Phi(x). \quad (109)$$

$G(x, y)$ satisfies the Klein-Gordon equation

$$(\square - m^2)G(x, y) = 0 \quad (110)$$

⁴The (homogeneous) Green function $G(x, y)$ is defined (as a distribution) on the whole of the product manifold $\mathcal{M} \times \mathcal{M}$, but in equation (106) we are restricting $G(x, y)$ to the subset $\mathcal{M} \times \Sigma$.

in either argument x or y and is antisymmetric: $G(x, y) = -G(y, x)$. Global hyperbolicity allows us to assume without loss of generality that the coordinates have been chosen so that Σ corresponds to the hypersurface $x^0 = t = 0$ or $y^0 = s = 0$. It will be helpful to be explicit and write $G(x, y) = G(t, \vec{x}; s, \vec{y})$. The equation (109) then reads

$$\Phi(t, \vec{x}) = \int_{\Sigma} \{ \pi(y) G(t, \vec{x}; 0, \vec{y}) - \phi(\vec{y}) \nabla_n^y G(t, \vec{x}; 0, \vec{y}) \} d\Sigma^y. \quad (111)$$

Let $\Phi(t, \vec{x})$ be the solution generated by data $\phi(\vec{y}), \pi(\vec{y})$, where $\phi(\vec{y})$ is an arbitrary smooth function of compact support, and $\pi(\vec{y}) = 0$. In this case, (111) is just

$$\Phi(t, \vec{x}) = \int_{\Sigma} -\phi(\vec{y}) \nabla_n^y G(t, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (112)$$

Evaluating this equation at $t = 0$, we find

$$\phi(\vec{x}) := \Phi(0, \vec{x}) = \int_{\Sigma} -\phi(\vec{y}) \nabla_n^y G(0, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (113)$$

However, this is exactly the definition of the Dirac delta distribution $\delta(\vec{x} - \vec{y})$. Differentiate (112):

$$\nabla_n^x \Phi(t, \vec{x}) = \int_{\Sigma} -\phi(\vec{y}) \nabla_n^x \nabla_n^y G(t, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (114)$$

Then,

$$0 = \pi(\vec{x}) = \nabla_n^x \Phi(0, \vec{x}) = - \int_{\Sigma} \phi(\vec{y}) \nabla_n^x \nabla_n^y G(0, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (115)$$

As $\phi(\vec{y})$ is arbitrary, we must have $\nabla_n^x \nabla_n^y G(0, \vec{x}; 0, \vec{y}) = 0$. Alternatively, let $\Phi(t, \vec{x})$ be the solution generated by data $\phi(\vec{y}), \pi(\vec{y})$, where $\pi(\vec{y})$ is an arbitrary smooth function of compact support, and $\phi(\vec{y}) = 0$. Then (111) reduces to

$$\Phi(t, \vec{x}) = \int_{\Sigma} \pi(y) G(t, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (116)$$

Evaluate this equation at $t = 0$:

$$0 = \phi(\vec{x}) = \Phi(0, \vec{x}) = \int_{\Sigma} \pi(y) G(0, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (117)$$

As $\pi(\vec{y})$ is arbitrary, we must have $G(0, \vec{x}; 0, \vec{y}) = 0$. Lastly, evaluating the normal derivative of Φ on Σ yields

$$\pi(\vec{x}) = \nabla_n^x \Phi(0, \vec{x}) = \int_{\Sigma} \pi(\vec{y}) \nabla_n^x G(0, \vec{x}; 0, \vec{y}) d\Sigma^y. \quad (118)$$

Collecting these results we conclude that for each fixed $\vec{y} \in \Sigma$, $G(t, \vec{x}; 0, \vec{y})$ is the unique distributional solution to the Klein-Gordon equation having data $\begin{pmatrix} 0 \\ \delta(\vec{x} - \vec{y}) \end{pmatrix}$, and $\nabla_n^y G(t, \vec{x}; 0, \vec{y})$ is the unique distributional solution to the Klein-Gordon equation having data $\begin{pmatrix} -\delta(\vec{x} - \vec{y}) \\ 0 \end{pmatrix}$. Define Θ_y and Δ_y as in (108). The completeness condition then follows from (109).

Thus, we may define a quantization procedure for any globally hyperbolic spacetime by first choosing a Cauchy surface and using (108) to expand the field in terms of distributional modes. A choice of positive-negative frequency would be made by defining the positive frequency field operator

$$2\Phi_{ppw}^+(0, x) := \phi(x) + i\pi(x). \quad (119)$$

The position Fock space would then be generated by the vacuum state, defined by

$$\Phi^+(t, \vec{x})|0\rangle = 0 \quad \forall (t, \vec{x}) \in \Sigma. \quad (120)$$

One may alternatively choose different encodings of the expansion coefficients into the positive frequency operator to define different vacua as desired. For example, in the case of a static spacetime with compact spatial sections, one

may Fourier decompose with respect to Killing time and use the encoding as in (30) to obtain a vacuum analogous to the Minkowski plane wave vacuum.

Canonical distributional modes were defined above for any choice of Cauchy surface in any globally hyperbolic spacetime, and a prescription for second quantization was given. In Section IV, distributional modes in Rindler space were constructed on the $\tau = 0$ hypersurface, but this construction is not a special case of the canonical construction. This is due to the fact that the unit normal vector field does not coincide with the timelike vector field corresponding to the (Rindler) coordinate time derivative used to construct the family $\{\Delta_\lambda\}$. This situation can be dealt with by relaxing the requirement that the unit normal derivative be used in the definition of Δ_λ . Instead, associate distributional modes with a choice of Cauchy surface *and* a choice of timelike vector field ξ , defining the modes Δ_λ as $\xi(\Theta_\lambda)$. For static spacetimes such as the Rindler spacetime, a natural choice of timelike vector field to use would be the Killing vector field. However, as far as the mathematical formalism is concerned, any timelike vector field can be used. Equation (109) would then be used to determine the expansion coefficients. Therefore, once a Cauchy surface and timelike derivative is chosen, and the distributional mode expansion made, second quantization may proceed as in the canonical case.

VI. DISCUSSION

In Minkowski space, our distributional modes correspond more naturally to pseudo plane waves rather than plane waves, and in particular are *not* Poincaré invariant. Nevertheless, we showed that pseudo plane waves can be used in place of plane waves to deduce important physical properties, such as the Rindler temperature. The fact that our distributional modes are related to the fundamental solution, which leads to our existence and uniqueness results, further suggests that pseudo plane waves may deserve further study in their own right.

At first sight, the distributional modes have a simple definition in terms of data on an initial data surface. In Rindler space, however, the definition of the distributional modes depended in a crucial way on Rindler coordinates. Although it turns out that the modes Θ_λ are invariantly defined on any globally hyperbolic spacetime, the modes Δ_λ were defined as the derivatives of Θ_λ with respect to the Rindler time coordinate, and therefore are not equivalent to their Minkowski counterparts. In this framework, therefore, the fundamental role of a family of observers appears as the choice of time coordinate used to differentiate the Θ modes to obtain the Δ modes.

Nonetheless, from the results of the theory of PDEs, we were able to produce a canonical generalization of the distributional modes. This construction most directly generalizes the Minkowski case in that the modes are defined using a Cauchy surface and the unit normal derivative, and then prescribing delta function data as in (108). Work is in progress on the generalization of the distributional modes for a given coordinate system and Cauchy surface.

A procedure based on canonical distributional modes for the second quantization of the scalar field in an arbitrary globally hyperbolic spacetime is readily generalized from the Minkowski procedure. There is no a priori need for Fourier transformation nor the need to find solutions explicitly other than the distributional mode solutions. Moreover, the canonical distributional mode solutions exist in any globally hyperbolic spacetime and have the same simple expression in terms of Cauchy data. To demonstrate the utility of this formalism, one should apply it to specific models, such as the trousers spacetime. Also desirable would be a rigorous treatment of the functional analysis involved in the formalism, as well as a detailed examination of how the formalism fits in with the many other approaches to curved space quantum field theory. These issues are all being actively pursued.

We owe a considerable debt to Corinne A. Manogue for inspiring this research with her idea of using the distributional modes *as modes*. A.A. would like to thank Robert Wald and Robert Geroch for helpful discussions and the Enrico Fermi Institute at the University of Chicago for hospitality. This work forms part of a dissertation [17] submitted to Oregon State University (by A.A.) in partial fulfillment of the requirements for the Ph.D. degree in mathematics, and was partially supported by NSF Graduate Research Fellowship 9616167-DGE.

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